FLUCTUATIONS OF MULTI-DIMENSIONAL KINGMAN-LÉVY PROCESSES

THU NGUYEN

Abstract. In the recent paper [15] we have introduced a method of studying the multi-dimensional Kingman convolutions and their associated stochastic processes by embedding them into some multi-dimensional ordinary convolutions which allows to study multi-dimensional Bessel processes in terms of the cooresponding Brownian motions. Our further aim in this paper is to introduce k-dimensional Kingman-Lévy (KL) processes and prove some of their fluctuation properties which are analoguous to that of k-symmetric Lévy processes. In particular, the Lévy-Itô decomposition and the series representation of Rosiński type for k-dimensional KL-processes are obtained.

Keywords and phrases: Cartesian products of Kingman convolutions; Rayleigh distributions

1. Introduction. notations and prelimilaries

The purpose of this paper is to introduce and study the multivariate KL processes defined in terms of multicariate Kingman convolutions. To begin with we review the following information of the Kingman convolutions and their Cartesian products.

Let $\mathcal{P} := \mathcal{P}(\mathbb{R}^+)$ denote the set of all probability measures (p.m.'s) on the positive half-line $\mathbb{R}^+$. Put, for each continuous bounded function f on $\mathbb{R}^+$,

$$(1)\quad \int_0^\infty f(x)\mu *_{1,\delta} \nu(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+\frac{1}{2})} \int_0^\infty \int_0^\infty \int_{-1}^1 f((x^2+2uxy+y^2)^{1/2})(1-u^2)^{s-1/2}\mu(dx)\nu(dy)du,$$

where μ and $\nu \in \mathcal{P}$ and $\delta = 2(s+1) \geq 1$ (cf. Kingman[7] and Urbanik[19]). The convolution algebra $(\mathcal{P}, *_{1,\delta})$ is the most important example of Urbanik convolution algebras (cf. Urbanik[19]). In language of the Urbanik convolution algebras, the *characteristic measure*, say σ_s, of the Kingman convolution has the Rayleigh density

$$(2)\qquad d\sigma_s(y) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} y^{2s+1} \exp\left(-(s+1)y^2\right)dy$$

with the characteristic exponent $\varkappa = 2$ and the kernel Λ_s

$$(3)\qquad \Lambda_s(x) = \Gamma(s+1)J_s(x)/(1/2x)^s,$$

where $J_s(x)$ denotes the Bessel function of the first kind,

$$(4)\qquad J_s(x) := \Sigma_{k=0}^\infty \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

Date: August 10, 2009.

It is known (cf. Kingman [7], Theorem 1), that the kernel Λ_s itself is an ordinary characteristic function (ch.f.) of a symmetric p.m., say F_s, defined on the interval [-1,1]. Thus, if θ_s denotes a random variable (r.v.) with distribution F_s then for each $t \in \mathbb{R}^+$,

$$\Lambda_s(t) = E\exp(it\theta_s) = \int_{-1}^{1} \cos(tx) dF_s(x). \tag{5}$$

Suppose that X is a nonnegative r.v. with distribution $\mu \in \mathcal{P}$ and X is independent of θ_s. The *radial characteristic function* (rad.ch.f.) of μ, denoted by $\hat{\mu}(t)$, is defined by

$$\hat{\mu}(t) = E\exp(itX\theta_s) = \int_0^\infty \Lambda_s(tx)\mu(dx), \tag{6}$$

for every $t \in \mathbb{R}^+$. The characteristic measure of the Kingman convolution $*_{1,\delta}$, denoted by σ_s, has the Maxwell density function

$$\frac{d\sigma_s(x)}{dx} = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} x^{2s+1} exp\{-(s+1)x^2\}, \quad (0 < x < \infty). \tag{7}$$

and the rad.ch.f.

$$\hat{\sigma}_s(t) = exp\{-t^2/4(s+1)\}. \tag{8}$$

Let $\tilde{P} := \tilde{\mathcal{P}}(\mathbb{R})$ denote the class of symmetric p.m.'s on $\mathbb{R}$. Putting, for every $G \in \mathcal{P}$,

$$F_s(G) = \int_0^\infty F_{cs} G(dc),$$

we get a continuous homeomorphism from the Kingman convolution algebra $(\mathcal{P}, *_{1,\delta})$ onto the ordinary convolution algebra $(\tilde{\mathcal{P}}, *)$ such that

$$F_s\{G_1 *_{1,\delta} G_2\} = (F_sG_1) * (F_sG_2) \qquad (G_1, G_2 \in \mathcal{P}) \tag{9}$$

$$F_s\sigma_s = N(0, 2s+1) \tag{10}$$

which shows that every Kingman convolution can be embedded into the ordinary convolution $*$.

Denote by $\mathbb{R}^{+k}, k = 1, 2, ...$ the k-dimensional nonnegative cone of $\mathbb{R}^k$ and $\mathcal{P}(\mathbb{R}^{+k})$ the class of all p.m.'s on $\mathbb{R}^{+k}$ equipped with the weak convergence. In the sequel, we will denote the multidimensional vectors and random vectors (r.vec.'s) and their distributions by bold face letters.

For each point z of any set A let δ_z denote the Dirac measure (the unit mass) at the point z. In particular, if $\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^{k+}$, then

$$\delta_{\mathbf{x}} = \delta_{x_1} \times \delta_{x_2} \times \ldots \times \delta_{x_k}, \quad (k\ times), \tag{11}$$

where the sign " $\times$ " denotes the Cartesian product of measures. We put, for $\mathbf{x} = (x_1, \cdots, x_k)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_k) \in \mathbb{R}^{+k}$,

$$\delta_{\mathbf{x}} \bigcirc_{s,k} \delta_{\mathbf{y}} = \{\delta_{x_1} \circ_s \delta_{y_1}\} \times \{\delta_{x_2} \circ_s \delta_{y_2}\} \times \cdots \times \{\delta_{x_k} \circ_s \delta_{y_k}\}, \quad (k\ times), \tag{12}$$

here and somewhere below for the sake of simplicity we denote the Kingman convolution operation $*_{1,\delta}, \delta = 2(s+1) \geq 1$ simply by $\circ_s, s \geq \frac{!}{2}$. Since convex combinations of p.m.'s of the form (11) are dense in $\mathcal{P}(\mathbb{R}^{+k})$ the relation (12) can be

extended to arbitrary p.m.'s $\mathbf{G}_1$ and $\mathbf{G}_2 \in \mathcal{P}(\mathbb{R}^{+k})$. Namely, we put

$$\mathbf{G}_1 \bigcirc_{s,k} \mathbf{G}_2 = \iint_{\mathbb{R}^{+k}} \delta_{\mathbf{x}} \bigcirc_{s,k} \delta_{\mathbf{y}} \mathbf{G}_1(d\mathbf{x})\mathbf{G}_2(d\mathbf{y}) \tag{13}$$

which means that for each continuous bounded function ϕ defined on $\mathbb{R}^{+k}$

$$\int_{\mathbb{R}^{+k}} \phi(\mathbf{z})\mathbf{G}_1 \bigcirc_{s,k} \mathbf{G}_2(d\mathbf{z}) = \iint_{\mathbb{R}^{+k}} \Big\{ \int_{\mathbb{R}^{+k}} \phi(\mathbf{z})\delta_{\mathbf{x}} \bigcirc_{s,k} \delta_{\mathbf{y}}(d\mathbf{z})\Big\} \mathbf{G}_1(d\mathbf{x})\mathbf{G}_2(d\mathbf{y}). \tag{14}$$

In the sequel, the binary operation $\bigcirc_{s,k}$ will be called *the k-times Cartesian product of Kingman convolutions* and the pair $(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_{s,k})$ will be called *the k-dimensional Kingman convolution algebra*. It is easy to show that the binary operation $\bigcirc_{s,k}$ is continuous in the weak topology which together with (1) and (13) implies the following theorem.

Theorem 1. *The pair $(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_{s,k})$ is a commutative topological semigroup with $\delta_{\mathbf{0}}$ as the unit element. Moreover, the operation $\bigcirc_{s,k}$ is distributive w.r.t. convex combinations of p.m.'s in $\mathcal{P}(\mathbb{R}^{+k})$.*

For every $\mathbf{G} \in \mathcal{P}(\mathbb{R}^{+k})$ the k-dimensional rad.ch.f. $\hat{\mathbf{G}}(\mathbf{t}), \mathbf{t} = (t_1, t_2, \cdots t_k) \in \mathbb{R}^{+k}$, is defined by

$$\hat{\mathbf{G}}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \Lambda_s(t_j x_j)\mathbf{G}(\mathbf{dx}), \tag{15}$$

where $\mathbf{x} = (x_1, x_2, \cdots x_k) \in \mathbb{R}^{+k}$. Let $\mathbf{\Theta_s} = \{\theta_{s,1}, \theta_{s,2}, \cdots, \theta_{s,k}\}$, where $\theta_{s,j}$ are independent r.v.'s with the same distribution F_s. Next, assume that $\mathbf{X} = \{X_1, X_2, ..., X_k\}$ is a k-dimensional r.vec. with distribution $\mathbf{G}$ and $\mathbf{X}$ is independent of $\mathbf{\Theta}_s$. We put

$$[\mathbf{\Theta}_s, \mathbf{X}] = \{\theta_{s,1}X_1, \theta_{s,2}X_2, ..., \theta_{s,k}X_k\}. \tag{16}$$

Then, the following formula is equivalent to (15) (cf. [14])

$$\widehat{\mathbf{G}}(\mathbf{t}) = Ee^{i<\mathbf{t},[\mathbf{\Theta}_s,\mathbf{X}]>}, \qquad (\mathbf{t} \in \mathbb{R}^{+k}). \tag{17}$$

The Reader is referred to Corollary 2.1, Theorems 2.3 & 2.4 [14] for the principal properties of the above rad.ch.f. Given $s \geq -1/2$ define a map $F_{s,k} : \mathcal{P}(\mathbb{R}^{+k}) \to \mathcal{P}(\mathbb{R}^k)$ by

$$F_{s,k}(\mathbf{G}) = \int_{\mathbb{R}^{+k}} (T_{c_1}F_s) \times (T_{c_2}F_s) \times \ldots \times (T_{c_k}F_s)\mathbf{G}(d\mathbf{c}), \tag{18}$$

here and in the sequel, for a distribution $\mathbf{G}$ of a r.vec. $\mathbf{X}$ and a real number c we denote by $T_c\mathbf{G}$ the distribution of $c\mathbf{X}$. Let us denote by $\tilde{\mathcal{P}}_{s,k}(\mathbb{R}^{+k})$ the sub-class of $\mathcal{P}(\mathbb{R}^k)$ consisted of all p.m.'s defined by the right-hand side of (18). By virtue of (15)-(18) one can prove the following theorem.

Theorem 2. *The set $\tilde{\mathcal{P}}_{s,k}(\mathbb{R}^{+k})$ is closed w.r.t. the weak convergence and the ordinary convolution $*$ and the following equation holds*

$$\hat{\mathbf{G}}(\mathbf{t}) = \mathcal{F}(F_{s,k}(\mathbf{G}))(\mathbf{t}) \qquad (\mathbf{t} \in \mathbb{R}^{+k}) \tag{19}$$

where $\mathcal{F}(\mathbf{K})$ denotes the ordinary characteristic function (Fourier transform) of a p.m. $\mathbf{K}$. Therefore, for any $\mathbf{G}_1$ and $\mathbf{G}_2 \in \mathbb{R}^{+k}$

$$F_{s,k}(\mathbf{G}_1) * F_{s,k}(\mathbf{G}_2) = F_{s,k}(\mathbf{G}_1 \bigcirc_{s,k} \mathbf{G}_2) \tag{20}$$

and the map $F_{s,k}$ commutes with convex combinations of distributions and scale changes $T_c, c > 0$. Moreover,

$$F_{s,k}(\Sigma_{s,k}) = N(\mathbf{0}, 2(s+1)\mathbf{I}) \tag{21}$$

where $\Sigma_{s,k}$ denotes the k-dimensional Rayleigh distribution and $N(\mathbf{0}, 2(s+1)\mathbf{I})$ is the symmetric normal distribution on $\mathbb{R}^k$ with variance operator $\mathbf{R} = 2(s+1)\mathbf{I}, \mathbf{I}$ being the identity operator. Consequently, Every Kingman convolution algebra $\big(\mathcal{P}(\mathbb{R}^{+k}), \bigcirc_{s,k}\big)$ is embedded in the ordinary convolution algebra $\big(\mathcal{P}_{s,k}(\mathbb{R}^{+k}), \star\big)$ and the map $F_{s,k}$ stands for a homeomorphism.

Let us denote by $\mathcal{E} = \{\mathbf{e} = (e_1, e_2, \ldots, e_k), e_j = \pm, j = 1, 2, \ldots, k\}$. It is convenient to regard the elements of $\mathcal{E}$ as sign vectors. Denote $\mathbb{R}^{+k}_{\mathbf{e}} = \{[\mathbf{e}, \mathbf{x}] : \mathbf{x} \in \mathbb{R}^{+k}\}$, where $[\mathbf{e}, \mathbf{x}] := (e_1 x_1, e_2 x_2, \ldots, e_k x_k)$. Then the family $\{\mathbb{R}^{+k}_{\mathbf{e}}, \mathbf{e} \in \mathcal{E}\}$ is a partition of the space $\mathbb{R}^k$. If $\mathbf{X}$ is a k-dimensional r.vec. with distribution $\mathbf{G}$, the k-symmetrization of $\mathbf{G}$, denoted by $\tilde{\mathbf{G}}$, is defined by

$$\tilde{\mathbf{G}} = \frac{1}{2^k} \sum_{\mathbf{e} \in \mathcal{E}} S_{\mathbf{e}} \mathbf{G}, \tag{22}$$

where the operator $S_{\mathbf{e}}$ is defined by

$$S_{\mathbf{e}}(\mathbf{x}) = [\mathbf{e}, \mathbf{x}] \qquad \mathbf{x} \in \mathbb{R}^k \tag{23}$$

and the symbol $S_{\mathbf{e}}\tilde{\mathbf{G}}$ denotes the image of $\mathbf{G}$ under $S_{\mathbf{e}}$.

Definition 1. *We say that a distribution $\mathbf{G} \in \mathcal{P}(\mathbb{R}^k)$ is k-symmetric, if the equation $\mathbf{G} = \tilde{\mathbf{G}}$ holds.*

Definition 2. *A p.m. $\mathbf{F} \in \mathcal{P}(\mathbb{R}^{+k})$ is called $\bigcirc_{s,k}-$infinitely divisible ($\bigcirc_{s,k}-$ID), if for every m=1, 2, ... there exists $\mathbf{F}_m \in \mathbf{P}(\mathbb{R}^{+k})$ such that*

$$\mathbf{F} = \mathbf{F}_m \bigcirc_{s,k} \mathbf{F}_m \bigcirc_{s,k} \ldots \bigcirc_{s,k} \mathbf{F}_m \quad (m \textit{ times}). \tag{24}$$

Definition 3. $\mathbf{F}$ *is called stable, if for any positive numbers a and b there exists a positive number c such that*

$$T_a\mathbf{F} \bigcirc_{s,k} T_b\mathbf{F} = T_c\mathbf{F} \tag{25}$$

By virtue of Theorem 2 it follows that the following theorem holds.

Theorem 3. *A p.m. $\mathbf{G}$ is $\bigcirc_{s,k} -ID$, resp. stable if and only if $H_{s,k}(\mathbf{G})$ is ID, resp. stable, in the usual sense.*

The following theorem gives a representation of rad.ch.f.'s of $\bigcirc_{s,k}-$ID distributions. The proof is a verbatim reprint of that for ([14], Theorem 2.6).

Theorem 4. *A p.m. $\mu \in ID(\bigcirc_{s,k})$ if and only if there exist a σ-finite measure M (a Lévy's measure) on $\mathbb{R}^{+k}$ with the property that $M(\mathbf{0}) = 0, \mathbf{M}$ is finite outside every neighborhood of $\mathbf{0}$ and*

$$\int_{\mathbb{R}^{+k}} \frac{\|\mathbf{x}\|^2}{1 + \|\mathbf{x}\|^2} \mathbf{M}(d\mathbf{x}) < \infty \tag{26}$$

and for each $\mathbf{t} = (t_1, ..., t_k) \in \mathbb{R}^{+k}$

$$-\log \hat{\mu}(\mathbf{t}) = \int_{\mathbb{R}^{+k}} \{1 - \prod_{j=1}^{k} \Lambda_s(t_j x_j)\} \frac{1 + \|\mathbf{x}\|^2}{\|\mathbf{x}\|^2} M(d\mathbf{x}), \tag{27}$$

where, at the origin $\mathbf{0}$*, the integrand on the right-hand side of (27) is assumed to be*

$$lim_{\|\mathbf{x}\|\to 0}\{1-\prod_{j=1}^{k}\Lambda_s(t_jx_j)\}\frac{1+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}=\Sigma_{j=1}^{k}\lambda_j^2t_j^2 \tag{28}$$

for nonnegative $\lambda_j, j = 1, 2, ..., k$*. In particular, if* $M = 0$*, then* μ *becomes a Rayleighian distribution with the rad.ch.f (see definition 4)*

$$-\log\hat{\mu}(\mathbf{t})=\frac{1}{2}\sum_{j=1}^{k}\lambda_j^2t_j^2,\quad \mathbf{t}\in\mathbb{R}^{+k}, \tag{29}$$

for some nonnegative $\lambda_j, j = 1, ..., k$*. Moreover, the representation (27) is unique.*

An immediate consequence of the above theorem is the following:

Corollary 1. *Each distribution* $\mu\in ID(\bigcirc_{s,k})$ *is uniquely determined by the pair* $[\mathbf{M},\boldsymbol{\lambda}]$*, where* $\mathbf{M}$ *is a Levy's measure on* $\mathbb{R}^{+k}$ *such that* $\mathbf{M}(\mathbf{0}) = 0$*,* $\mathbf{M}$ *is finite outsite every neighbourhood of* $\mathbf{0}$ *and the condition (26) is satisfied and* $\boldsymbol{\lambda} := \{\lambda_1, \lambda_2, \cdots \lambda_k\} \in \mathbb{R}^{+k}$ *is a vector of nonnegative numbers appearing in (29). Consequently, one can write* $\mu \equiv [\mathbf{M}, \boldsymbol{\lambda}]$*.*

In particular, if $\mathbf{M}$ *is zero measure then* $\mu = [\boldsymbol{\lambda}]$ *becomes a Rayleighian p.m. on* $\mathbb{R}^{+k}$ *as defined as follows:*

Definition 4. *A k-dimensional distribution, say* $\boldsymbol{\Sigma}_{s,k}$*, is called a Rayleigh distribution, if*

$$\boldsymbol{\Sigma}_{s,k}=\sigma_s\times\sigma_s\times\cdots\times\sigma_s\quad (k\ times). \tag{30}$$

Further, a distribution $\mathbf{F}\in\mathcal{P}(\mathbb{R}^{+k})$ *is called a Rayleighian distribution if there exist nonnegative numbers* $\lambda_r, r = 1, 2\cdots k$ *such that*

$$\mathbf{F}=\{T_{\lambda_1}\sigma_s\}\times\{T_{\lambda_2}\sigma_s\}\times\ldots\times\{T_{\lambda_k}\sigma_s\}. \tag{31}$$

It is evident that every Rayleigh distribution is a Rayleighian distribution. Moreover, every Rayleighian distribution is $\bigcirc_{s,k}-$ID. By virtue of (7) and (30) it follows that the k-dimensional Rayleigh density is given by

$$g(\mathbf{x})=\Pi_{j=1}^{k}\frac{2^k(s+1)^{k(s+1)}}{\Gamma^k(s+1)}x_j^{2s+1}exp\{-(s+1)||\mathbf{x}||^2\}, \tag{32}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^{+k}$ and the corresponding rad.ch.f. is given by

$$\hat{\Sigma}_{s,k}(\mathbf{t})=Exp(-|\mathbf{t}|^2/4(s+1)),\quad \mathbf{t}\in\mathbb{R}^{+k}. \tag{33}$$

Finally, the rad.ch.f. of a Rayleighian distribution $\mathbf{F}$ on $\mathbb{R}^{+k}$ is given by

$$\hat{\mathbf{F}}(\mathbf{t})=Exp(-\frac{1}{2}\sum_{j=1}^{k}\lambda_j^2t_j^2) \tag{34}$$

where $\lambda_j, j = 1, 2, \ldots, k$ are some nonnegative numbers.

2. Multivariate Bessel processes

3. Multivrariate Kingman-Lévy processes and their Lévy-Itô decomposition

Suppose that $\mu_t, t \geq 0$ is continuous semigroup in $ID(\bigcirc_{s,k})$, that is for any $t, s \geq 0$

$$\mu_t \bigcirc_{s,k} \mu_s = \mu_{t+s} \tag{35}$$

and $\{\mu_t\}$ is continuous at 0 i.e.

$$lim_{t\to 0}\mu_t = \delta_0.$$

By virtue of Theorem 2 it follows that $\{\mathcal{F}_{s,k}(\mu_t)\}$ is an ordinary continuous convolution semigroup on $\mathbb{R}^k$. Putting, for each $\mathbf{x} \in \mathbb{R}^{k+}$ and for every Borel subset $\mathcal{E}$ of $\mathbb{R}^{k+}$,

$$\mathbf{P}(t, \mathcal{E}, \mathbf{x}) = \mu_t \bigcirc_{s,k} \delta_{\mathbf{x}}(\mathcal{E}) \tag{36}$$

and using the rad.ch.f. it follows that the family $\{\mathbf{P}(t, \mathcal{E}, \mathbf{x}), t \geq 0\}$ satisfies the Chapman-Kolmogorov equation and, consequently, the formula (36) defines transition probabilities of a $\mathbb{R}^{k+}-$valued homogeneous strong Markov Feller process $\{\mathbf{X}_t^{\mathbf{x}}, t \geq 0\}$, say, such that it is stochastically continuous and has a cadlag version (compare [11], Theorem 2.6).

Definition 5. *A $\mathbb{R}^{k+}$-valued stochastic process $\{\mathbf{X}_t, t \geq 0\}$ is called a Kingman-Lévy process, if $\mathbf{X}_t =$*
(i) $\mathbf{X}_0 = \mathbf{0}$ $\quad (P.1)$;
(ii) There exists a $\mathbb{R}^{k+}-$valued homogeneous strong Markov Feller process having a cadlag version $\{\mathbf{X}_t^{\mathbf{x}}, t \geq 0\}$ with transition probabilities defined by (36) and $\mathbf{X}_t = \mathbf{X}_t^{\mathbf{0}}, t \geq 0$;

4. Fluctuations of Multidimensional Bessel Processes

Definition 6. *Let $(W_t, t \geq 0)$ be a d-dimensional Brownian motion (d=1, 2, ...). The Euclidean norm of (W_t), denoted by $B_t, t \geq 0$ is called a Bessel process.*

It has been proved that Bessel processes inherit the well-known characteristics of Brownian motions: They are independent stationary "increments" processes with continuous sample paths. The term 'increment' is defined as follows:

Definition 7. *For any $s > u$ the random variable $|W_s - W_u|$ is called an increments of the Bessel process.*

The following theorem gives a Lévy-Khinczyn representation of the Bessel process in the sense of the Kingman convolution.

Theorem 5. *The radial characteristic function $\phi(x)$ of the Bessel process (B_t) is of the form*

$$\phi(x) = exp\{-\frac{tx^2}{4(s+1)}\} \qquad x, t \geq 0 \tag{37}$$

where d=2(s+1).

Since for any $s > u$ the 'increment' of the Bessel process (B_t) is infinitely divisible in the ordinary convolution $*$ we have the following representation of the Fourier transform of B_{s-u}.

$$\mathcal{F}_{B_{s-u}}(x) = exp(-(s-u)\psi(x)) \tag{38}$$

where $\psi(x)$ is a symmetric characteristic exponent

$$\psi(x) = \frac{1}{2}\sigma^2 + \int_0^\infty (1 - cos\, xv)\Pi(dv) \tag{39}$$

where the measure Π satisfies the condition
begin equation

$$\int_0^\infty (min(1, x^2)\Pi(dx) < \infty. \tag{40}$$

which implies the following Lévy-Itô decomposition.

Theorem 6. *(Lévy-Itô decomposition) There exists a Brownian motion $X_t^{(1)}$ and a compound Poison process $X_t^{(2)}$ independent of $X_t^{(1)}$ such that*

$$B_t = ||W_t|| \stackrel{d}{=} X_t^{(1)} + X_t^{(2)} \qquad (t \geq 0). \tag{41}$$

Before stating the Wienner-Hopf factorization (WHf) theorem for Bessel processes we introduce some concepts and notations. The importance of WHf is that it gives us information of the ascending and descending ladder processes. We begin by recalling that for $\alpha, \beta \geq 0$ the Laplace exponents $\kappa(\alpha, \beta)$ and $\hat{\kappa}(\alpha, \beta)$ of the ascending ladder process $(\hat{L}^{-1}, \hat{H})$ and the descending ladder process $(\hat{L}^{-1}, \hat{H})$. Further, we define

$$\bar{G}_t = sup\{s < t : \bar{X}_s = X_s\} \text{ and } \underline{G}_t = sup\{s < t : \underline{X}_t = X_s.$$

Theorem 7. *(Wienner-Hopf Factorization) Let $(B_t, t \geq 0)$ be a Bessel process. Denote by $\mathbf{e}_p$ an independent and exponentially distributed random variable.*

The pairs $(\bar{G}_{\mathbf{e}_p}, \bar{X}_{\mathbf{e}_p})$ and $(\mathbf{e}_p - \bar{G}_{\mathbf{e}_p}, \bar{X}_{\mathbf{e}_p} - X_{\mathbf{e}_p})$ are independent and infinitely divisible, yielding the factorization

$$\frac{p}{p - i\nu + \psi(\theta)} = \Psi^+(\nu, \theta).\Psi^-(\nu, \theta) \qquad \nu, \theta \in \mathbb{R}, \tag{42}$$

ψ^+, ψ^- being Fourier transforms and called the Wienner-Hopf factors.

5. Levy-Ito decomposition of Kingman-Levy processes

References

[1] Bingham, N.H., Random walks on spheres, Z. Wahrscheinlichkeitstheorie Verw. Geb., **22**, (1973), 169-172.

[2] Bingham, N.H., On a Theorem of Klosowska about generalized convolutions, Colloquium Math., **28** No. **1**, (1984), 117-125.

[3] Cox, J.C., Ingersoll, J.E.Jr., and Ross, S.A., A theory of the term structure of interest rates. Econometrica, **53**(2), (1985).

[4] Feller, W., An Introduction to probability Theory and Its Applications, John Wiley & Sons Inc., Vol.II, 2nd Ed., (1971).

[5] Ito, K., Mckean H.P., Jr., Diffusion processes and their sample paths, Berlin-Heidelberg-New York. Springer (1996).
[6] Kalenberg O., Random measures, 3rd ed. New York: Academic Press, (1983).
[7] Kingman, J.F.C., Random walks with spherical symmetry, Acta Math., **109**, (1963), 11-53.
[8] Kyprianou, Andreas E., Introductory lectures on fluctuations of Lévy processes with applications,
[9] Levitan B.M., Generalized translation operators and some of their applications, Israel program for Scientific Translations, Jerusalem, (1962).
[10] Linnik Ju. V., Ostrovskii, I. V., Decomposition of random variables and vectors, Translation of Mathematical Monographs, vol. 48, American Mathematical Society, Providence R. L, 1977, ix+380 pp.,$38.80. (Translated from the Russian, 1972, by Israel Program for Scientific Translations).
[11] Nguyen V.T, Generalized independent increments processes, Nagoya Math. J.**133**, (1994), 155-175.
[12] Nguyen V.T., Generalized translation operators and Markov processes, Demonstratio Mathematica, **34** No 2, ,295-304.
[13] Nguyen T.V., OGAWA S., Yamazato M. A convolution Approach to Mutivariate Bessel Processes, Proceedings of the 6th Ritsumeikan International Symposium on "*Stochastic Processes and Applications to Mathematical Finance*", edt. J. Akahori, S. OGAWA and S. Watanabe, World Scientific, (2006) 233-244.
[14] Nguyen V. T., A Kingman convolution approach to Bessel processes, Probab. Math. Stat, Probab. Math. Stat. **29**, fasc. 1(2009) 119-134.
[15] Nguyen V. T., An analogue of the Cramér-Lévy theorem for multi-dimensional Rayleigh distributions, arxiv.org/abs/0907.5035.
[16] Revuz, D. and Yor, M., Continuous martingals and Brownian motion. Springer-verlag Berlin Heidelberg, (1991).
[17] Sato K, Lévy processes and infinitely divisible distributions, Cambridge University of Press, (1999).
[18] Shiga T., Watanabe S., Bessel diffusions as a one-parameter family of diffusion processes, Z. Warscheinlichkeitstheorie Verw. geb. 27,(1973), 34-46.
[19] Urbanik K., Generalized convolutions, Studia math., **23** (1964), 217-245.
[20] Urbanik K., Cramér property of generalized convolutions,Bull. Polish Acad. Sci. Math.**37** No 16 (1989), 213-218.
[21] Vólkovich, V. E., On symmetric stochastic convolutions, J. Theor. Prob. **5**, No. **3**(1992), 417-430.

Department of Mathematics; International University, HCM City; No.6 Linh Trung ward, Thu Duc District, HCM City; Email: nvthu@hcmiu.edu.vn